1.6 The zoo of basic analytic functions, their derivatives, and branches for their inverses. (We'll continue section 1.6 on Monday.)

Def If $f : \mathbb{C} \to \mathbb{C}$ is analytic on all of $\mathbb{C}$, then f is called *entire.*

Examples:

- $f(z) = z^n,\ n \in \mathbb{Z} \setminus \{0\}$ $\qquad f'(z) = n z^{n-1}$

- $f(z) = e^z$ $\qquad f'(z) = e^z$

$f(z) = \cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$ $\qquad f'(z) = -\sin z$!

$$f'(z) = \frac{1}{2}\left(e^{iz} i + e^{-iz}(-i)\right) = \frac{i}{2}\left(e^{iz} - e^{-iz}\right) = -\sin z$$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \qquad \frac{1}{i} = -i$$

$f(z) = \sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$ $\qquad f'(z) = \cos z$

$$\tan z = \frac{\sin z}{\cos z} \qquad (\tan z)' = \sec^2 z \ \ldots$$

$$\sec z = \frac{1}{\cos z}$$

Here is a non-entire function, but you can define it as a differentiable function locally, or using any branch domain for $\log z$: $= \ln|z| + i\arg z \quad (+i2\pi k)$

- $f(z) = z^a := e^{a\log(z)}$, $a \in \mathbb{C}$ — for given local choice of $\arg z$

$\boxed{f'(z) = a z^{a-1}}$?

check $f'(z) = e^{a\log z}\dfrac{a}{z} = a\dfrac{e^{a\log z}}{e^{\log z}} = a e^{(a-1)\log z} = a z^{a-1}$

same arg choice

Question: For $f(z) = z^a$ as above, does the multi-value definition agree with $f(z) = z^n$, $n \in \mathbb{Z}$?

motivation
$z = e^{\log z}$
$z^a \underset{\text{should}}{=} (e^{\log z})^a = e^{a\log z}$

$n \in \mathbb{N}$
old def $z^n = \left(|z|e^{i\arg z}\right)^n = |z|^n e^{in\arg z}$
(also z^{-n}, $n \in \mathbb{N}$)

new def
$$z^n = e^{n\log z} = e^{n(\ln|z| + i\arg z + i2\pi k)}$$
$$= \underbrace{e^{n\ln|z|}\, e^{in\arg z}}_{\text{old } z^n}\, \underbrace{e^{i2\pi kn}}_{1}$$
✓

Question: For $f(z) = z^a$ as above, does the multi-value definition agree with the multivalue definition of the n^{th} root function $f(z) = z^{\frac{1}{n}}$, $n \in \mathbb{N}$?

old def $z^{1/n} = |z|^{1/n} e^{\frac{i}{n}(\arg z + 2\pi k)}$ $\quad k \in \mathbb{Z}$
$= |z|^{1/n} e^{\frac{i}{n}\arg z} e^{\frac{i2\pi k}{n}}$, $k = 0, 1, \ldots n-1$
yields the n n^{th} roots

new def: $z^{\frac{1}{n}} = e^{\frac{1}{n}\log z}$
$= e^{\frac{1}{n}(\ln|z| + i\arg z + 2\pi k)}$
$= |z|^{\frac{1}{n}} e^{\frac{i}{n}(\arg z + 2\pi k)}$

Same

So also $z^{\frac{m}{n}}$ old, new defs agree

Math 4200
Wednesday September 16

1.6 differentiation and mapping of elementary functions and branches of their inverses, and compositions of all of these.

Announcements: We'll begin by covering the part of Monday's notes which introduces section 1.6, before proceding into today's notes which discuss how to find *branched domains* (aka *fundamental domains*) on which multi-valued functions can be defined as single-valued analytic functions.

quiz today §1.5.

$f(z) = e^z$ range omits $\{0\}$; one point

Branches of analytic functions overview: If f is *entire,* i.e. analytic on all of $\mathbb{C}$, then it turns out (Picard's Theorem) that if f is *not* a constant function, then the range of f omits no more than two points in $\mathbb{C}$! Furthermore, it turns out that the zeroes of $f'(z)$ are isolated (i.e. if $f'(z_0) = 0$ then there exists $r > 0$ such that $f'(z) \neq 0 \ \forall \ w \in D(z_0, r) \setminus \{z_0\}$.) So f has a local inverse function except at possibly a countable set of $z \in \mathbb{C}$.

In most cases this means one can construct a differentiable partial "inverse" function g on a *very large* subdomain of $\mathbb{C}$. It will satisfy half of the inverse function condition, namely

e.g. $e^{\log z} = z$

$$f(g(z)) = z.$$

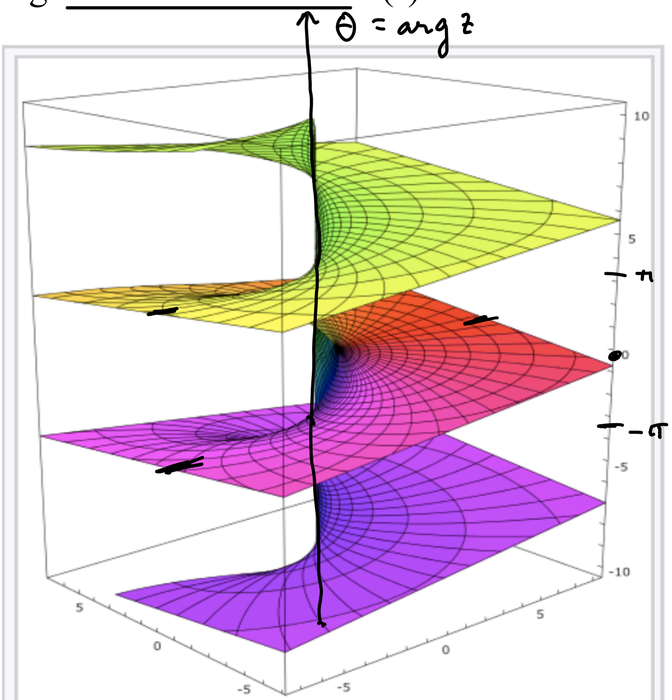
And the domain of g can usually be chosen to be a connected open domain $A \subseteq \mathbb{C}$ with a just finite number of curves removed from $\mathbb{C}$ to get A. These omitted curves are called *branch cuts*, and the choice of (partial) inverse function is called a *branch* of the inverse function. Branch cuts always terminate either at ∞ (which means $|z| \to \infty$), or at finite points, and these are called *branch points*.

In our text section 1.6 these branch domains are called *fundamental domains*. There is usually some freedom in how they are chosen.

$g(w) = \log w$

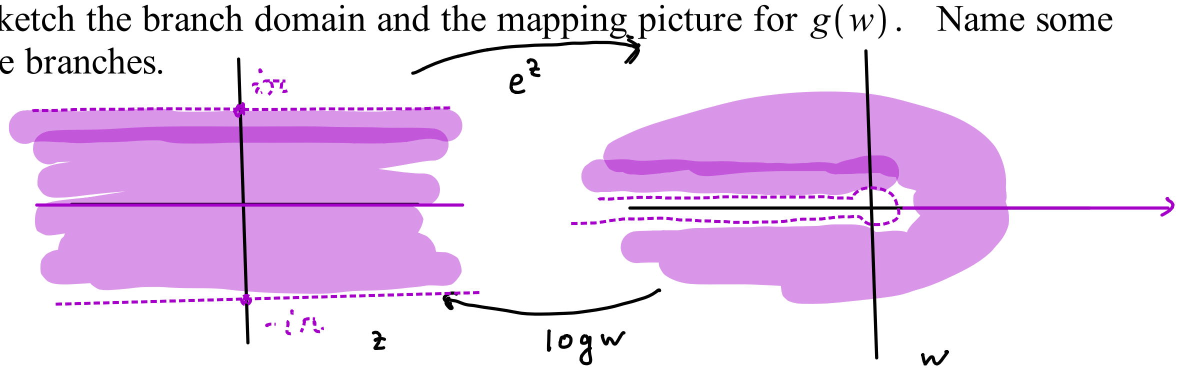
The most central example of this discussion is $f(z) = e^z$ which omits only the point 0 in its range, and branch choices for the multivalued inverse $\log(z)$. A nice graphic picture from the wikipedia page on the complex logarithm which visualizes the possible branch choices for $\log(z)$, is obtained by plotting the parametric surface $(r\cos(\theta), r\sin(\theta), \theta)$ in $\mathbb{R}^3$. I haven't figured out precisely what the curves on the helicoid are, although they seem to be related to some conformal parameterization of the helicoid, not the $r-\theta$ one. Since the helicoid is a *minimal surface*, i.e. locally area minimizing and a possible shape for soap films, it turns out that it can be parameterized in a conformal way using harmonic functions. (!)

$-\infty < \theta < \infty$



A plot of the multi-valued imaginary part of the complex logarithm function, which shows the branches. As a complex number z goes around the origin, the imaginary part of the logarithm goes up or down. This makes the origin a branch point of the function.

Example 1) $f(z) = e^z$, $g(w) = \log w = \ln|w| + i\,arg\, w$ where we choose $-\pi < arg(w) < \pi$. This is called the standard branch of $\log w$. We've seen this before, but sketch the branch domain and the mapping picture for $g(w)$. Name some other possible branches.

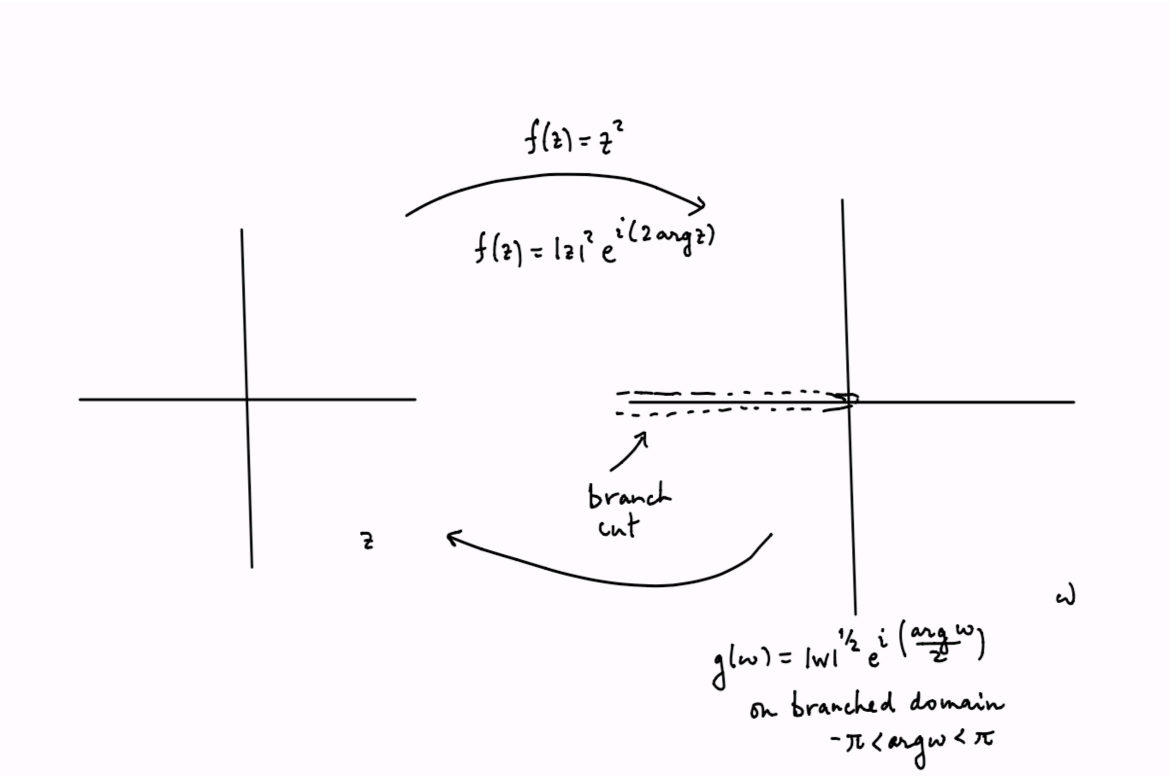


Example 2) $f(z) = z^2$, $g(w) = \sqrt{w}$ (for some branch choice). Note for any branch choice of g,

On Friday we'll do this in depth

$$f(g(w)) = w$$
$$f'(g(w))g'(w) = 1$$
$$g'(w) = \frac{1}{f'(g(w))} = \frac{1}{2\,g(w)} = \frac{1}{2}w^{-\frac{1}{2}}.$$

Describe the range of the branch of the square root function defined below. Write down two other branch choices - one using the same branch cut, and another one using a different cut.



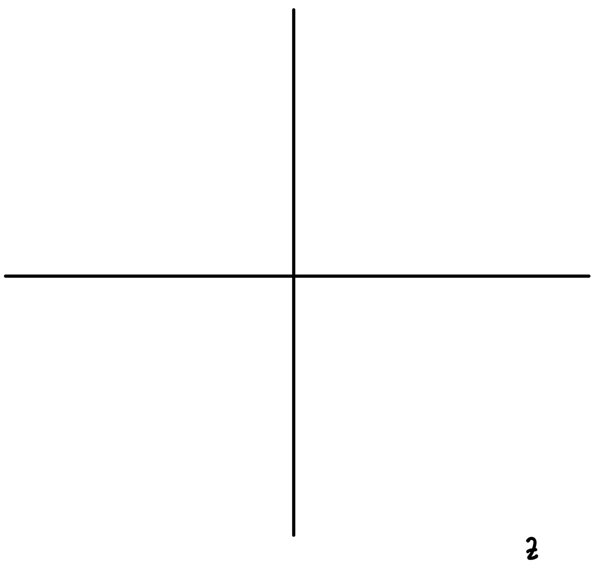
Example 3) Find a definition and branched domain for

$$f(z) = \sqrt{z^2 - 1}\,.$$

(In your homework for next week you will do an analogous procedure for $g(z) = \sqrt{z^3 - 1}$.) Begin by identifying branch points based on where f or f' cannot be not defined as an analytic function.

Then

a) Writing $f(z) = \sqrt{z^2 - 1} = \sqrt{z-1}\,\sqrt{z+1}$ leads to one possible way of proceding.



b) Considering f as a composition, $f(z) = g \circ h(z)$ with $h(z) = z^2 - 1$ and $g(w) = \sqrt{w}$ recovers the first branched domain, but also leads to a choice with only a finite branch cut, as well as the original one.